Far Dissipation Range of Turbulence

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Abstract

The very small scales of isotropic, Navier-Stokes turbulence at Reynolds number $\mathcal{R}_\lambda \approx 15$ are studied by high-resolution direct numerical simulation (DNS) and by integration of the direct-interaction (DIA) equations. The DNS follows the tail of the energy spectrum over more than thirty decades of magnitude. The energy spectrum in the far-dissipation range $5k_d < k < 10k_d$ is well-fitted by $k^\alpha \exp(-ck/k_d)$, where k_d is the Kolmogorov dissipation wavenumber, $\alpha \approx 3.3$ and $c \approx 7.1$. For values of m that emphasize the far-dissipation range, the fields $(-\nabla^2)^m \mathbf{u}$ exhibit strong spatial intermittency, associated with gentle spatial variations of the lower-k part of the velocity field. DIA analysis gives a prefactor k^3 and an exponential decay more rapid than DNS. Averaging over an ensemble of DIA solutions, suggested by the observed intermittency, removes some of the discrepancy.

The smallest scales of incompressible, isotropic Navier-Stokes turbulence are associated with the far-dissipation range of wavenumbers $k \gg k_d$, where $k_d = \epsilon^{1/4}\nu^{-3/4}$ is the Kolmogorov dissipation wavenumber, ϵ is the rate of dissipation of hydrodynamic kinetic energy per unit mass and ν is kinematic viscosity. Most of the dissipation takes place at $k < k_d$. The wavenumbers $k \gg k_d$ have attracted attention for a number of years. There has been controversy concerning the asymptotic form of the energy spectrum as $k \to \infty$. The smallest scales are of further interest because they display strong intermittency even at Reynolds numbers so low that there is no basis for a fractal cascade.

It is reasonable to assume that the wavenumbers $k \gg k_d$ represent spectral tails of flow structures of spatial scale $\geq 1/k_d$. An analogy is the exponential spectral tail of a shock that obeys Burgers' equation. There are other possibilities. One is that very-high-wavenumber excitation comes mostly from exceptionally strongly strained regions that give rise to observed exponential-like skirts of the probability distribution function (pdf) of vorticity.

A number of authors have discussed kinetic energy spectra for $k \gg k_d$ of the form

$$E(k) \propto f(k/k_d) \exp[-c(k/k_d)^n] \tag{1}$$

where c is a constant, f is a weak function of k/k_d and $1 \leq n \leq 2$ [1–7]. The direct interaction approximation (DIA), a perturbative treatment, gives $n = 1$ and $f \propto (k/k_d)^3$ [2]. Perturbation approximation can be justified for $k \gg k_d$ because the mode amplitudes are very nearly the linear response, under molecular viscosity, to quadratic forcing by modes of lower k [8]. The additional, unjustified assumption in the DIA analysis is that the statistics of wave-vector amplitudes for $k \gg k_d$ are nearly Gaussian. The plausible effect of the intermittency actually present in the far-dissipation range is to enhance mean nonlinear transfer and thereby raise the spectrum level above that predicted by DIA.

Foias, Manley and Sirovich [3] have shown that $n \geq 1$, under certain assumptions of smoothness of the velocity field in a finite box. In view of this inequality, The DIA results suggest that $n = 1$ is exact for a finite box. The particular form

$$E(k) \propto k^{\alpha} \exp(-ck/k_d) \tag{2}$$

seems consistent with a body of experimental and computer data [7].

Strong intermittency in the far-dissipation range at modest Reynolds numbers was predicted some years ago on the basis of a simple physical argument [8]: $E(k)$ falls off steeply for $k \gg k_d$. Consequently, fluctuation, on spatial macroscales, of parameters like k_d in (1) yields spatial intermittency at scales $O(1/k)$ that increases without limit as $k/k_d \to \infty$.

An ongoing computation project has achieved resolutions up to 512^3 (wavenumber range $1 \leq k \leq 256$) on a CM-200 computer at Los Alamos National Laboratory [9]. The direct numerical simulation (DNS) described in this paper is limited by arithmetic precision rather than its resolution of 256^3 (wavenumber range $1 \leq k \leq 128$). A nominal steady state was maintained by forcing confined to $k < 3$, at a level determined to give the desired value of the Taylor microscale Reynolds number $\mathcal{R}_\lambda$. The Taylor microscale λ for an isotropic turbulent flow is a length defined by $\lambda = (15\nu v_0^2/\epsilon)^{1/2}$, where v_0^2 is the mean-square velocity in any direction, and $\mathcal{R}_\lambda \equiv v_0\lambda/\nu$.

The solid line in Fig. 1 represents the time-averaged wavenumber spectrum of kinetic energy $E(k)$ for a run with $\nu = 0.026$, $\mathcal{R}_\lambda \approx 14.9$, $k_d \approx 9.65$. The very close fit of the

high-k part of the spectrum to a straight line (exponential decay) is apparent. Fig. 2 shows $k\,d\ln E(k)/dk$ vs k for this data. If $E(k)$ has the form (2), this plot is a straight line whose slope is $-c/k_d$ and whose intercept on the vertical axis is α. The straight line in Fig. 2 is a least-squares fit to the data over the range $50 \leq k \leq 100$; it gives $c \approx 7.1$, $\alpha \approx 3.3$. Equation (2) seems well supported. The confidence level for α is not high, because it is not certain that the wavenumber range is long enough to give strictly asymptotic results. It cannot even be asserted that $f(k/k_d)$ is exactly a power. However, the value $n = 1$ in (1) does seem strongly favored.

A previous study [7] gave a negative value for α. The range used for fitting in [7] was $0.5k_d \leq k \leq 3k_d$, which is too low to give asymptotic behavior. Note that in Fig. 2 the data points curve downward at small k. An intercept $\alpha = 0$ corresponds to tangency at $k = 30 \approx 3k_d$ [10].

The dotted lines in Fig. 1 show the spectra in three subregions of the cyclic box, defined by the x-space filter $\exp[-|\mathbf{x} - \mathbf{x}_c|^2/(L/32)^2]$, where $L = 2\pi$ is the box size and $\mathbf{x}_c$ is the subregion center. The nominal linear dimension of a subregion is thus $L/16$. In order to sufficiently reduce errors from chopping, the cyclic box was repeated in each direction to give a total of 3^3 replicas before the filtering. The filtered field was transformed to k-space and subjected to solenoidal projection before the spectrum was computed. An effect of the filtering is marked depression of spectrum level for $k \lesssim 16$.

We have found that the distribution of spectral slopes over a set of 64 subregions, evenly spaced in the cyclic box, is consistent with the picture of intermittency [8] in which the parameters describing the spectral tail are slowly-varying functions of spatial position. The three subregion spectra plotted in Fig. 1 are those with minimum, median and maximum values of $E(k = 60)$. Note that the maximum and minimum values of $E(k = 60)$ differ by a factor of over 10^{10}, despite the relatively small difference in slope.

Fig. 3 shows pdf's of the differentiated velocity fields $(-\nabla^2)^m u_i$, averaged over the three components $i = 1, 2, 3$. Note the marked increase of intermittency with m. Fig. 4 is a visualization of the field $\nabla^8 u_1$. The shaded surfaces are where the absolute value of field amplitude equals twice its root-mean-square value. The evident three-dimensionality of these regions suggests intermittency that is associated with gentle spatial variation on principal dissipation scales rather than with exceptional regions that are strongly strained into thin sheets or tubes. The flatness $\langle(\nabla^8 u_1)^4\rangle/\langle(\nabla^8 u_1)^2\rangle^2$ is 57. This large value is associated with the sharp peak of the pdf at zero amplitude, rather than with the broad skirts at large amplitude values. The latter represent probabilities too small to affect low-order statistics.

The spectral support of the field $\nabla^8 u_1$ is effectively confined to the range $15 < k < 40$. A field with spectral support confined to $50 < k < 100$, the region where $E(k)$ is accurately proportional to $k^\alpha \exp(-ck/k_d)$, can be constructed by applying the filter $\exp[-(k-75)^2/200]$ to the wave-vector transform of $u_1(\mathbf{x})$ and transforming back to x space. Fig. 5 shows the regions where the absolute value of the amplitude of this field exceeds twice its root-mean-square value. The field $\nabla^{24} u_1$ has approximately the same spectral support, but it is too noisy to give clean visualizations.

The similarity between Figs. 4 and 5 is striking. The regions of high intensity are in the same locations and have similar shape, but are smaller in Fig. 5. This behavior suggests that the spectral support of the very small scales represents the spectral tail of larger structures. Similar behavior is exhibited by repeated differentiation of the velocity profile of a Burgers

shock.

We have integrated the DIA equations with the same forcing and viscosity as in the DNS. The dashed line in Fig. 1 shows the result for $E(k)$. The DIA spectrum tail falls within the range of DNS subregion values, but below the median. There are two obvious causes for discrepancy between DIA and DNS. One is the depression of high-k energy transfer in DIA by sweeping effects in response functions [2]. At $k = 75$, the sweeping decorrelation frequency $v_0 k$ is about 30% of the viscous decay frequency νk^2. Two-point closures that are invariant to random Galilean transformation do not display the strong sweeping decorrelation at large k [11].

The second cause is the observed intermittency, which is not captured by DIA. Consider the mean spectrum obtained by averaging over a finite-size ensemble of DIA solutions, or other closure solutions, with different values of spectrum level in the forcing region. As $k \to \infty$, this mean spectrum is dominated by the ensemble member with the smallest value of c/k_d. Thus the effective value of c/k_d is decreased by the averaging. The prefactor exponent $\alpha = 3$ is unchanged by the averaging. This latter value is a property of a broad family of two-point closures [7]. The averaging over closure predictions is not explored further here, but it remains possible that simple statistical distributions of the input parameters to the closure can give satisfactory approximations both to $E(k)$ and to the large flatness factors of the fields $(-\nabla^2)^m \mathbf{u}$.

The procedure of averaging over solutions with different parameter values has an analytical implication that does not depend on closure approximation. Suppose that the box containing the flow is of infinite size, but with spectral support confined to wavenumbers above some wavenumber k_0. A local wave-vector analysis centered on an arbitrary point $\mathbf{x}_c$ can be carried out by applying the filter $\exp(-k_0^2 |\mathbf{x} - \mathbf{x}_c|^2)$ to the velocity field and performing solenoidal projection on the result.

Suppose that the spectral tail thereby associated with a region of the box with dimensions $O(1/k_0)$ has the asymptotic behavior (2), where k_d is now a weak function of $\mathbf{x}_c$. If any value of $k_d(\mathbf{x}_c)$, however large, occurs for some $\mathbf{x}_c$, then averaging over the entire infinite box will give something slower than exponential decay of the spectrum as $k \to \infty$. For example, if the pdf of k_d over the box were $\propto \exp(-k_d^2/k_c^2)$ as $k_d \to \infty$, where k_c is a constant parameter, then averaging of (2) over this pdf would give a full-box spectrum $E(k) \propto \exp[-(2ck/k_c)^{2/3}]$ as $k \to \infty$, apart from an algebraic prefactor. Finally, if the size of the box were made finite but $\gg 1/k_0$, then this full-box spectrum would change over from stretched-exponential to simple-exponential decay at some wavenumber $\gg k_c$. The bound obtained by Foias, Manley and Sirovich [3] assumes a finite box.

REFERENCES

[1] A. A. Townsend, Proc. Roy. Soc. London A **208**, 534 (1951).

[2] R. H. Kraichnan, J. Fluid Mech. **5**, 497 (1959).

[3] C. Foias, O. Manley, and L. Sirovich, Phys. Fluids A **2**, 464 (1990).

[4] L. M. Smith and W. C. Reynolds, Phys. Fluids A **3**, 992 (1991).

[5] O. Manley, Phys. Fluids A, **4**, 1320 (1992).

[6] A. J. Domaradzki, Phys. Fluids A, **4**, 2037 (1992).

[7] S. Kida, R. H. Kraichnan, R. S. Rogallo, F. Waleffe, and Y. Zhou, preprint (1992).

[8] R. H. Kraichnan, Phys. Fluids **10**, 2081 (1967).

[9] S. Chen, G. D. Doolen, R. H. Kraichnan, Z. S. She, Phys. Fluids A, **5**, xxxx (1993).

[10] J. A. Domaradzki (private communication) has found that $\alpha > 0$ is favored by fits to several data sets in the range $k > 2k_d$.

[11] R. H. Kraichnan, Phys. Fluids **9**, 1728 (1966).

FIGURE CAPTIONS

FIG. 1.   Linear-log plot of $E(k)$ vs. k. Solid line, DNS spectrum; dotted lines, subregional spectra with largest, median and smallest values of $E(k = 60)$; dashed line, DIA spectrum.

FIG. 2.   The function $k \, d \ln E(k)/dk$ vs k for the DNS spectrum. The straight line is a least-squares fit to the data points for $50 \leq k \leq 100$.

FIG. 3.   Pdf $P(w)$ of the field $w = (-\nabla^2)^m u_i$ for $m = 0$ (dashed), $m = 2$ (dotted) and $m = 4$ (solid), averaged over $i = 1, 2, 3$.

FIG. 4.   Perspective view of the surface where $|\nabla^8 u_1|$ equals twice its root-mean-square value.

FIG. 5.   Perspective view of the surface where the absolute value of the filtered field with spectral support centered at $k = 75$ equals twice its root-mean-square value.